Correction: Chasing a Fast Robber on Planar Graphs and Random Graphs

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Abstract

Consider a variant of the Cops and Robber game, in which the robber has unbounded speed, i.e., can take any path from her vertex in her turn, but she is not allowed to pass through a vertex occupied by a cop. Let $c_\infty(G)$ denote the number of cops needed to capture the robber in a graph G in this variant, and let $\mathrm{tw}(G)$ denote the treewidth of G. In [1, Theorem 1] we showed that if G is planar then $c_\infty(G) = \Theta(\mathrm{tw}(G))$, and there is a polynomial-time constant-factor approximation algorithm for computing $c_\infty(G)$. One part of the argument, namely the proof of $c_\infty(G) = \Omega(\mathrm{tw}(G))$, was incomplete. Here we give a complete proof for this statement.

We will need a few definitions. An *apex graph* is a graph H that has a vertex v such that $H - v$ is planar. For two undirected simple graphs G and H, we say G *contains H as a contraction* if H can be obtained by applying a sequence of edge contractions to G. We say G is *H-minor-free* if H is not a subgraph of any contraction of G. For any positive integer r, let Γ_r be the graph obtained from the triangulated $r \times r$ grid by joining a degree-2 corner to all the boundary vertices. See Figure 1 for an illustration. Formally, we have

$$V(\Gamma_r) = \{(i, j) : 1 \le i, j \le r\},$$

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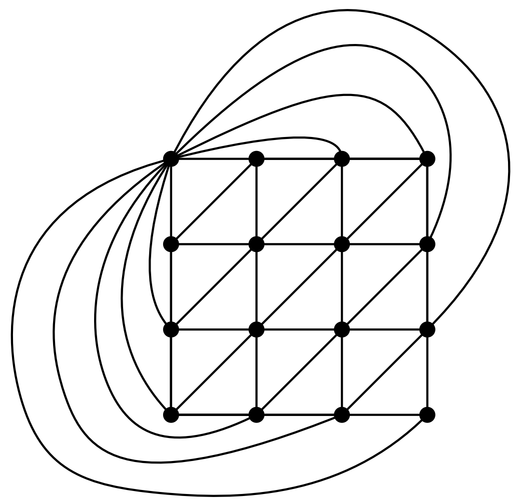

Figure 1: The graph Γ_4

and

$$\begin{aligned} E(\Gamma_r) = \{&(i,j)(i',j') : |i-i'|+|j-j'|=1\} \\ &\cup \{(i,j)(i+1,j+1) : 1 \leq i \leq r-1, 1 \leq j \leq r-1\} \\ &\cup \{(1,r)(i,j) : i \in \{1,r\} \text{ or } j \in \{1,r\}\} \setminus \{(1,r)(1,r)\}. \end{aligned}$$

Fomin, Golovach, and Thilikos [2, Theorem 3] proved that for every apex graph H, there exists $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction. We will prove that any H-minor-free G has $c_\infty(G) \geq \mathrm{tw}(G)/(3c_H) - 1$. Since any planar graph is K_5-minor-free, and K_5 is an apex graph, this implies $c_\infty(G) = \Omega(\mathrm{tw}(G))$ for any planar graph G.

Let G be an H-minor-free graph, and let $r = \lfloor \mathrm{tw}(G)/c_H \rfloor$. By [2, Theorem 3], G contains Γ_r as a contraction. It is easy to see that contracting an edge does not help the robber, since she has unbounded speed, and it does not hurt the cops. Therefore, $c_\infty(G) \geq c_\infty(\Gamma_r)$, thus it suffices to show that $c_\infty(\Gamma_r) \geq (r-2)/3$.

Consider the graph Γ_r. A *nonboundary row* is a subset of vertices of the form $\{(i,j) : 2 \leq j \leq r-1\}$ for some $i \in \{2,\ldots,r-1\}$. A *nonboundary column* is defined similarly. Note that there are exactly $r-2$ nonboundary rows and $r-2$ nonboundary columns. Let A be a nonboundary row and B be a nonboundary column. Then $A \cup B$ is called a *cross*. Note that for any two crosses C and C', the subset $C \cup C'$ induces a connected subgraph of Γ_r.

For a set $S \subseteq V(\Gamma_r)$, denote by $\overline{N}(S)$ the set of vertices that are in S or have a neighbour in S. We claim that, for any subset $S \subseteq V(\Gamma_r)$ of size smaller than $(r-2)/3$, there is a cross C with $\overline{N}(S) \cap C = \emptyset$. For any $s \in V(R)$, observe that $\overline{N}(\{s\})$ intersects at most 3 nonboundary rows. Since $|S| < (r-2)/3$, the set $\overline{N}(S)$ intersects less than $r-2$ nonboundary rows. So there exists a nonboundary row A not intersecting $\overline{N}(S)$. Symmetrically, there exists a nonboundary column B not intersecting $\overline{N}(S)$. Thus the cross $A \cup B$ does not intersect $\overline{N}(S)$, and this proves the claim.

Suppose there are less than $(r-2)/3$ cops in the game. We give a strategy for the (fast) robber such that, whenever the cops are in a subset S, the robber is at some vertex of a cross C with $\overline{N}(S) \cap C = \emptyset$. In the beginning, such a cross exists by the claim, and the robber may position herself accordingly. Suppose at the beginning of a round, the cops are in a subset S, and the robber is at some vertex of a cross C with $\overline{N}(S) \cap C = \emptyset$. Then the cops move to a new subset S'. Since $S' \subseteq \overline{N}(S)$, we have $S' \cap C = \emptyset$, so the robber is not captured in this round. Moreover, by the claim, there exists a cross C' with $\overline{N}(S') \cap C' = \emptyset$. Since S' does not intersect $C \cup C'$, and $C \cup C'$ induces a connected subgraph, there is a cop-free path from the robber's current position to some vertex in C', and the robber may move there. Using this strategy the robber will never be caught, and the proof is complete.

References

[1] N. Alon and A. Mehrabian, Chasing a fast robber on planar graphs and random graphs, J Graph Theory 78(2) (2015), 81–96.

[2] F. V. Fomin, P. Golovach, and D. M. Thilikos, Contraction obstructions for treewidth, Journal of Combinatorial Theory, Series B 101 (2011), 302–314.