Nowhere-zero Unoriented Flows in Hamiltonian Graphs

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Abstract

An *unoriented flow* in a graph, is an assignment of real numbers to the edges, such that the sum of the values of all edges incident with each vertex is zero. This is equivalent to a flow in a bidirected graph all of whose edges are extraverted. A *nowhere-zero unoriented k-flow* is an unoriented flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$. It has been conjectured that if a graph has a nowhere-zero unoriented flow, then it admits a nowhere-zero unoriented 6-flow. We prove that this conjecture is true for hamiltonian graphs, with 6 replaced by 12.

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1 Introduction

Let G be a directed graph. A *nowhere-zero flow* in G is an assignment of non-zero real numbers to edges of G such that for every vertex, the sum of the values of incoming edges is equal to sum of the values of outgoing edges. A *nowhere-zero k-flow* is a nowhere-zero flow such that the assigned values are integers with absolute values less than k.

A celebrated conjecture of Tutte says that:

Tutte's 5-flow Conjecture [8]. *Every bridgeless graph admits a nowhere-zero 5-flow.*

Note that the assumption that G is bridgeless is necessary, since if G has a bridge then it does not admit a nowhere-zero flow. Jaeger showed that every bridgeless graph admits a nowhere-zero 8-flow, see [5]. Next, Seymour proved that every bridgeless graph admits a nowhere-zero 6-flow [7]. For a thorough account of the above conjecture and subsequent results, see [4] and [10].

Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. The *incidence matrix* of G, denoted by $W(G)$, is an $n \times m$ matrix defined as

$$W(G)_{i,j} = \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i, \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The flows of G are indeed the elements of the null space of $W(G)$. If $[c_1, \ldots, c_m]^T$ is an element of the null space of $W(G)$, then we can assign value c_i to e_i and consequently obtain a flow. Therefore, in the language of linear algebra, Tutte's 5-flow Conjecture says that if G is a directed bridgeless graph, then there exists a vector in the null space of $W(G)$, whose entries are non-zero integers with absolute value less than 5.

One may also study the elements of null space of the incidence matrix of an undirected graph. For an undirected graph G, the incidence matrix of G, $W(G)$, is defined as follows:

$$W(G)_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ and } v_i \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

An element of the null space of $W(G)$ is a function $f : E(G) \longrightarrow \mathbb{R}$ such that for all vertices $v \in V(G)$ we have

$$\sum_{u \in N(v)} f(uv) = 0,$$

where $N(v)$ denotes the set of adjacent vertices to vertex v. We call such a function an *unoriented flow* on G, as contrasted to the usual "oriented

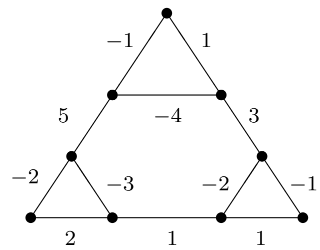


Figure 1: A graph with a nowhere-zero unoriented 6-flow.

flow". An *unoriented k-flow* is an unoriented flow whose values are integers with absolute value less than k. Figure 1 shows a nowhere-zero unoriented 6-flow.

There is a conjecture for unoriented flows similar to the Tutte's 5-flow Conjecture for oriented flows:

Let G be an undirected graph with incidence matrix W. If there exists a vector in the null space of W whose entries are non-zero real numbers, then there also exists a vector in that space, whose entries are non-zero integers with absolute value less than 6, or equivalently,

Zero-Sum Conjecture [1]. *If G is a graph with a nowhere-zero unoriented flow, then G admits a nowhere-zero unoriented 6-flow.*

Using the graph given in Figure 1, one can see that 6 can not be replaced with 5. It is known that for $d > 2$ any d-regular graph admits a nowhere-zero unoriented 7-flow (see [2]).

A *bidirected graph (oriented sign graph)* is a graph with each edge oriented as one of the four possibilities: $\bullet\!-\!\blacktriangleleft\!-\!\blacktriangleright\!-\!\bullet$, $\bullet\!-\!\blacktriangleright\!-\!\blacktriangleright\!-\!\bullet$, $\bullet\!-\!\blacktriangleright\!-\!\blacktriangleleft\!-\!\bullet$, $\bullet\!-\!\blacktriangleleft\!-\!\blacktriangleleft\!-\!\bullet$. An edge of the first type is called *extraverted*, an edge of the third type is called *introverted*, and an edge of the second or fourth type is called *directed*. Let G be a bidirected graph. For any $v \in V(G)$, the set of all edges with tails (resp. heads) at v is denoted by $E^+(v)$ (resp. $E^-(v)$). Function $f : E(G) \longrightarrow \mathbb{R}$ is a *bidirected flow* of G if for every $v \in V(G)$, we have

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).$$

If f takes values from the set $\{0, \pm 1, \ldots, \pm(k-1)\}$, then it is called a *bidirected k-flow.*

Bidirected flows generalize the concepts of oriented and unoriented flows. For if we orient all edges of G to be directed, then a bidirected flow in this setting corresponds to a usual (oriented) flow, and if we orient all edges of

G to be extraverted, then a bidirected flow corresponds to an unoriented flow.

In 1983, Bouchet proposed the following conjecture.

Bouchet's Conjecture [3, 9]. *Every bidirected graph that has a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow.*

Bouchet proved that his conjecture is true with 6 replaced by 216. Then Zyka in [11] proved that conjecture is true with 6 replaced by 30. In [6] Khelladi showed that if G is a 4-connected graph, then conjecture is true with 6 replaced by 18. Also in [9], it was shown that Bouchet's Conjecture is true for every 6-edge connected graph.

In this language, the Zero-Sum Conjecture says that if we orient all edges of G to be extraverted, and the obtained bidirected graph has a nowhere-zero bidirected flow, then it also admits a nowhere-zero bidirected 6-flow. Interestingly, this is equivalent to the Bouchet's Conjecture.

Theorem [2]. *Bouchet's Conjecture and the Zero-Sum Conjecture are equivalent.*

This theorem shows that the Zero-Sum Conjecture is true with 6 replaced by 30. The goal of this paper is to prove that the Zero-Sum Conjecture is true for hamiltonian graphs, with 6 replaced by 12.

Let G be an undirected hamiltonian graph. We obtain a bidirected graph by letting all edges of G to be extraverted. In this paper we will see that if the obtained graph has a nowhere-zero bidirected flow then it admits a nowhere-zero bidirected 12-flow. Equivalently, we show that if G has a nowhere-zero unoriented flow then it admits a nowhere-zero unoriented 12-flow. If G has certain special properties, stronger results are proved. In the next Section some definitions and basic results on unoriented flows are discussed. In Section 3, we prove the claim for graphs with an odd number of vertices, and in Section 4, for graphs with an even number of vertices.

2 Preliminaries

Let $G = (V, E)$ be a graph. The number of vertices of G and the number of edges of G are called the *order* and the *size* of G, respectively. The graph G is called *even* (resp. *odd*), if its size is even (resp. odd). An *even vertex* (resp. *odd vertex*) of G is a vertex of even (resp. odd) degree. A *circuit* in G is a closed walk with no repeated edge. The graph G is called *Eulerian* if all of its vertices are even, or equivalently, if each of its connected components is a circuit. We write $H \subseteq G$ if H is a subgraph of G. Let G_1, G_2 be subgraphs of G. The subgraph of G with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ is denoted by $G_1 \cup G_2$.

Let $T \subseteq E(G)$. The subgraph *induced by* T, denoted by $G[T]$, is a subgraph of G whose edge set is T and whose vertex set consists of the endpoints of the edges in T. For a vertex $v \in V(G)$, we denote the number of edges in T incident to v by $deg_T(v)$. Let $f : E(G) \longrightarrow \mathbb{R}$. The *support* of f, is the set of edges on which f is non-zero, and is denoted by $\mathrm{supp}(f)$. A cycle C of G is called *a Hamilton cycle*, if it contains all vertices of G. The graph G is called *hamiltonian* if it has a Hamilton cycle.

The following theorem determines all graphs having a nowhere-zero unoriented flow.

Theorem A [1]. *Let G be a connected graph. Then the following hold:*
(i) *If G is bipartite, then G admits a nowhere-zero unoriented flow if and only if it is bridgeless.*
(ii) *If G is not bipartite, then G admits a nowhere-zero unoriented flow if and only if removing any of its edges does not make any bipartite connected component.*

Theorem A together with Seymour's 6-flow theorem [7] have the following corollary.

Corollary A [1]. *If a bipartite graph has a nowhere-zero unoriented flow, then it admits a nowhere-zero unoriented 6-flow.*

The following easy Lemma will be useful.

Lemma A [2]. *Every even circuit admits a nowhere-zero unoriented 2-flow.*

3 Hamiltonian Graphs of Odd Orders

In this section we prove that if G is a hamiltonian graph of odd order with a nowhere-zero unoriented flow, then G admits a nowhere-zero unoriented 12-flow. The following lemma and its corollary play a key role in our proofs.

Lemma 3.1. *Let G_1, G_2 be subgraphs of graph G with $G = G_1 \cup G_2$. If G_1 has a nowhere-zero unoriented k_1-flow and G_2 has a nowhere-zero unoriented k_2-flow, then G admits a nowhere-zero unoriented k_1k_2-flow.*

Proof. Let $g_i : E(G_i) \longrightarrow \mathbb{R}$ be a nowhere-zero unoriented k_i-flow in G_i, for $i = 1, 2$. Define the functions $f_i : E(G) \longrightarrow \mathbb{R}$ such that $f_i = g_i$ on $E(G_i)$ and $f_i = 0$, elsewhere. Let $f = f_1 + k_1 f_2$. We claim that f is a nowhere-zero unoriented k_1k_2-flow in G. For each $u \in V(G)$, we have:

$$\sum_{v \in N(u)} f(uv) = \sum_{v \in N(u)} f_1(uv) + k_1 \sum_{v \in N(u)} f_2(uv) = 0.$$

Furthermore, for each $e \in E(G)$, if $f_2(e) \neq 0$, then we have

$$|f(e)| = |f_1(e) + k_1 f_2(e)| \geq k_1|f_2(e)| - |f_1(e)| \geq k_1 - (k_1 - 1) > 0.$$

If $f_2(e) = 0$, then $e \in E(G_1)$ and $|f(e)| = |f_1(e)| > 0$. Also we have

$$|f(e)| = |f_1(e) + k_1 f_2(e)| \leq (k_1 - 1) + k_1(k_2 - 1) = k_1k_2 - 1 < k_1k_2.$$

Thus f is a nowhere-zero unoriented k_1k_2-flow in G and the proof is complete. $\square$

Using a simple induction we get the following corollary.

Corollary 3.1. *Let $G_1, \dots, G_m$ be subgraphs of graph G with the property that $G = \bigcup_{i=1}^{m} G_i$. If G_i has nowhere-zero unoriented k_i-flow for $i = 1, 2, \dots, m$, then G admits a nowhere-zero unoriented $\prod_{i=1}^{m} k_i$-flow.*

Throughout this section we have the following assumptions: Graph G is a hamiltonian graph of odd order, $C = (v_1, \dots, v_n, v_1)$ is a Hamilton cycle, and H is the subgraph induced by $E(G) \backslash E(C)$.

Lemma 3.2. *There exist Eulerian subgraphs K_1 and K_2 of G containing H, such that K_1 is even and K_2 is odd. Moreover, if H is connected, then K_1 and K_2 are connected.*

Proof. If H has no odd vertex, then it is Eulerian and since the parity of degrees of vertices of H and G are the same, G is Eulerian, too. On the other hand, sizes of G and H have different parities. So if H is even, let $K_1 = H$ and $K_2 = G$. Otherwise, G is even and we can put $K_1 = G$ and $K_2 = H$.

So, suppose that H has $2m$ odd vertices. Let $1 \leq a_1 < a_2 < \cdots < a_{2m} \leq n$ be their indices. Suppose that H is odd (the proof for the even case is similar). For each $i = 1, \dots, 2m - 1$, let

$$E_i = \{v_{a_i}v_{a_i+1}, v_{a_i+1}v_{a_i+2}, \dots, v_{a_{i+1}-1}v_{a_{i+1}}\},$$

and let

$$E_{2m} = \{v_{a_{2m}}v_{a_{2m}+1}, \dots, v_{n-1}v_n, v_nv_1, \dots, v_{a_1-1}v_{a_1}\}.$$

Note that the sets E_i decompose $E(C)$, thus $\sum_{i=1}^{2m} |E_i| = n$.

Now, let $A_1 = E_1 \cup E_3 \cup \cdots \cup E_{2m-1}$ and $A_2 = E_2 \cup E_4 \cup \cdots \cup E_{2m}$. Since $\sum_{i=1}^{2m} |E_i|$ is odd, either A_1 or A_2 has odd cardinality. Without loss of generality assume that $|A_1|$ is odd.

Let $K_i = G[E(H) \cup A_i]$ for $i = 1, 2$. Since both $E(H)$ and A_1 have odd cardinality and they are disjoint, K_1 is even. Moreover, every vertex of K_1 has even degree, because after adding the set of edges A_1 to H, the degree of every odd vertex increases by 1, and the degree of every even vertex does not change or increases by 2. Hence K_1 is an even Eulerian subgraph containing H. It can be similarly proved that K_2 is an odd Eulerian subgraph containing H.

Note that K_1 and K_2 have been obtained by adding some paths to H, such that the endpoints of these paths are in H. Thus if H is connected, then so are K_1 and K_2. The proof is complete. $\square$

Theorem 3.1. *If H is connected and non-bipartite, then G admits a nowhere-zero unoriented 4-flow.*

Proof. Let C' be an odd cycle of H, and $K = C' \cup C$. Since K is the union of two edge-disjoint cycles, every vertex of K is even and since $E(C')$ and $E(C)$ have odd cardinalities, K is even. Also since C is a Hamilton cycle, K is connected. Thus K is an even circuit and it has a nowhere-zero unoriented 2-flow by Lemma A.

By Lemma 3.2, there exists a connected even Eulerian subgraph K_1 of G containing H. Thus K_1 is an even circuit and admits a nowhere-zero unoriented 2-flow. We have $K \cup K_1 = G$, so the theorem follows from Lemma 3.1. $\square$

Theorem 3.2. *Suppose G be has a nowhere-zero unoriented flow. If H is connected, then G admits a nowhere-zero unoriented 8-flow.*

Proof. If H is not bipartite, then by Theorem 3.1, G admits a nowhere-zero unoriented 4-flow. Otherwise, let X and Y be the two parts of H. Since G is a non-bipartite graph with a nowhere-zero unoriented flow, by Theorem A, removing each edge of G makes no bipartite connected component. Thus there exist at least two edges $uv, u'v' \in E(C)$ such that u, v are in the same part of H, and also u', v' are in the same part of H.

Since H is connected and bipartite, there exist even paths P and P' in H with endpoints u, v and u', v', respectively. Let $C_1 = P \cup (C \backslash \{uv\})$ and $C_2 = P' \cup (C \backslash \{u'v'\})$. The graphs C_1 and C_2 are even circuits, and so they have nowhere-zero unoriented 2-flows. Note that $C \subseteq C_1 \cup C_2$.

By Lemma 3.2, there exists a connected even Eulerian subgraph K_1 of G containing H. Thus K_1 is an even circuit and admits a nowhere-zero unoriented 2-flow. This follows that

$$G \subseteq C \cup H \subseteq C_1 \cup C_2 \cup K_1.$$

Furthermore, each of C_1, C_2, K_1 admits a nowhere-zero unoriented 2-flow. Hence by Corollary 3.1, G admits a nowhere-zero unoriented 8-flow. $\square$

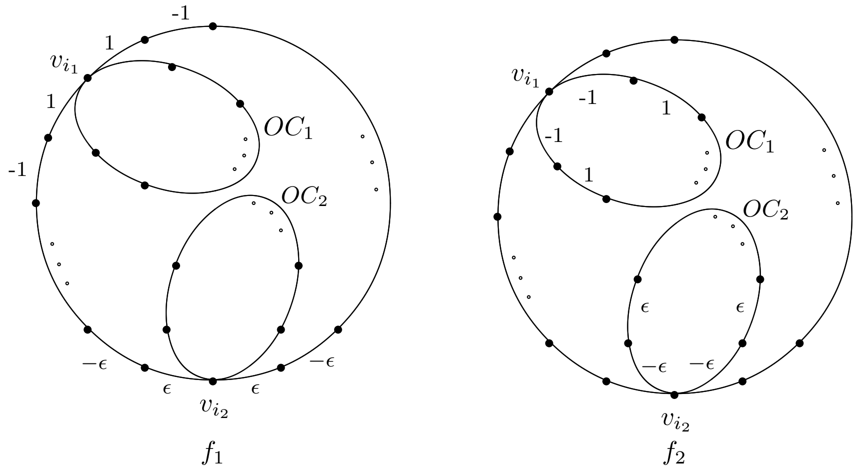


Figure 2

Lemma 3.3. *There exists a subgraph K of G that admits a nowhere-zero unoriented 3-flow and contains H.*

Proof. By Lemma 3.2 there is an odd Eulerian subgraph K_2 of G that contains $E(H)$. Let $OC_1, OC_2, \ldots, OC_t$, and $EC_1, EC_2, \ldots, EC_s$ be the odd connected components and even connected components of K_2, respectively. The subgraph K_2 is Eulerian, so each of these components is a circuit. Since K_2 is odd, t is odd. Let v_{i_j} be an arbitrary vertex of OC_j, for $j = 1, \ldots, t$. Without loss of generality assume that $i_1 \leq \cdots \leq i_t$.

Now, we define two functions f_1 and f_2 on $E(G)$, in the following way: Let $OC_j = (v_{i_j}, u_{j,1}, \ldots, v_{i_j})$, for $j = 1, \ldots, t$. Let P_j be the path in C from v_{i_j} to $v_{i_{j+1}}$, taking indices module t. Define f_1 to alternately take values $+1$ and -1 within each P_j, but not to alternate at each v_{i_j}. Define $f_2|_{E(OC_j)}$ to have $f_2(v_{i_j} u_{j,1}) = -f_1(v_{i_j - 1} v_{i_j})$ and to be alternating from there on. See Figure 2, where ϵ is either $+1$ or -1 depending on the parity of $i_2 - i_1$. Define f_2 to be alternating on every EC_k for $k = 1, 2, \ldots, s$.

Now, define function f on $E(G)$ by $f(e) = f_1(e) + f_2(e)$. It can be seen that $\operatorname{supp}(f_1) = E(C)$, $\operatorname{supp}(f_2) = E(K_2)$ and for every vertex $v \in V(G)$, we have $\sum_{u \in N(v)} f(uv) = 0$. Furthermore $|f(e)| = |f_1(e) + f_2(e)| < 3$.

On the other hand, for every $e \in E(H)$, since $e \notin \operatorname{supp}(f_1)$, we have $f(e) = f_1(e) + f_2(e) = f_2(e)$, which is non-zero because $E(H) \subseteq E(K_2)$. Note that f may be zero on some edges of C. Let $K = G[\operatorname{supp}(f)]$. Hence f is a nowhere-zero unoriented 3-flow in K, we have $E(H) \subseteq E(K)$, and the proof is complete. □

Theorem 3.3. *Suppose that G has a nowhere-zero unoriented flow and that H is non-bipartite. Then G admits a nowhere-zero unoriented 6-flow.*

Proof. Let C' be an odd cycle in H, and $K = C \cup C'$. Clearly, K is connected, so it is an even circuit and admits a nowhere-zero unoriented 2-flow. By Lemma 3.3, there exists a subgraph K' that admits a nowhere-zero unoriented 3-flow and $E(H) \subseteq E(K')$. Therefore $G = K \cup K'$ and the theorem follows from Lemma 3.1. $\square$

Lemma 3.4. *Suppose that G has a nowhere-zero unoriented flow. Then for every $i = 1, \ldots, n$, there exists an edge $v_j v_k \in E(G)$ ($v_j v_k \neq v_i v_{i+1}$), such that the path from v_j to v_k along C, containing $v_i v_{i+1}$, has odd length.*

Proof. Define two subsets of $V(G)$ as

$$X = \{v_t \mid \text{ the path from } v_t \text{ to } v_i \text{ along } C \backslash \{v_i v_{i+1}\} \text{ has even length } \},$$

and $Y = V(G) \backslash X$. Since G is a non-bipartite graph with a nowhere-zero unoriented flow, by Theorem A, $G \backslash \{v_i v_{i+1}\}$ is not bipartite, so X or Y contains two adjacent vertices. Let v_j and v_k be these vertices. Note that v_i and v_{i+1} are not adjacent in $G \backslash \{v_i v_{i+1}\}$, so $v_j v_k \neq v_i v_{i+1}$. We deduce from the definitions of X and Y that the path from v_j to v_k along C, not containing $v_i v_{i+1}$, has even length. Since n is odd, the other path from v_j to v_k along C is odd and we are done. $\square$

Let $v_j v_k$ be the edge given in Lemma 3.4, and P be the path from v_j to v_k along C containing $v_i v_{i+1}$. So $P \cup \{v_j v_k\}$ is an even cycle. We call such a cycle a *good cycle*.

Lemma 3.5. *Suppose that G has a nowhere-zero unoriented flow. If there exist good cycles $C_1, C_2, \ldots, C_t$ such that each edge of C is contained either in exactly one of the given cycles, or in exactly two consecutive cycles (considering C_t, C_1 as consecutive cycles), then there exists a subgraph K' with a nowhere-zero unoriented 4-flow such that $E(C) \subseteq E(K')$.*

Proof. Define $K' = \bigcup_{i=1}^{t} C_i$. Suppose that $C_1 = (v_{i_1}, v_{i_1+1}, \ldots, v_{j_1}, v_{i_1})$. Define function f_1 on $E(G)$ as follows: $f_1(v_{i_1} v_{i_1+1}) = -1, f_1(v_{i_1+1} v_{i_1+2}) = +1, \ldots, f_1(v_{j_1-1} v_{j_1}) = -1, f_1(v_{j_1} v_{i_1}) = +1$, and let $f_1 = 0$ on $E(G) \backslash E(C_1)$. For each $i = 2, \ldots, t$, similarly define f_i on $E(C_i)$ (alternately -1 and $+1$) in such a way that f_i is equal to f_{i-1} on $E(C_{i-1}) \cap E(C_i)$, and is zero on $E(G) \backslash E(C_i)$. So $\mathrm{supp}(f_i) = E(C_i)$ for $i = 1, \ldots, t$.

Now, define function f on $E(G)$ as follows:

$$f(e) = \sum_{i=1}^{t-1} f_i(e) + 2 f_t(e).$$

For every vertex v and each i, $\sum_{u \in N(v)} f_i(uv) = 0$. This implies that $\sum_{u \in N(v)} f(uv) = 0$. For every edge e, since e is contained in at most two good cycles, $|f(e)| \leq 1 + 2 \times 1 < 4$. In addition, if $e \in C_i \cap C_{i+1}$ for some i, $1 \leq i \leq t-1$, then the values of f_i and f_{i+1} on e are the same. Thus $f(e) \neq 0$. If $e \in C_t \cap C_1$, then $|f(e)| = |2f_t(e) + f_1(e)| \geq 2|f_t(e)| - |f_1(e)| > 0$. Therefore, f is a nowhere-zero unoriented 4-flow in K', and we are done.□

Theorem 3.4. *Let G be a hamiltonian graph of odd order with a nowhere-zero unoriented flow. Then G admits a nowhere-zero unoriented 12-flow.*

Proof. Let $C = (v_1, \ldots, v_n, v_1)$ be a Hamilton cycle of G, and H be the subgraph induced by $E(G) \backslash E(C)$. According to Lemma 3.3, there exists a subgraph K that admits a nowhere-zero unoriented 3-flow and $E(H) \subseteq E(K)$. If we prove the existence of a subgraph K' having a nowhere-zero unoriented 4-flow, such that $E(C) \subseteq E(K')$, then the theorem is proved by Lemma 3.1.

By Lemma 3.4, every edge $e \in E(C)$ is contained in some good cycle. So there exist coverings of C with good cycles. Let $\{C_i\}_{i \in I}$ be a covering of C with good cycles such that $|I|$ is minimum. We claim that any edge $e \in E(C)$ is contained in at most two of the C_i. Take any $e = v_j v_{j+1}$. Let $I_e = \{i \in I \mid e \in C_i\}$ and $P_e = \bigcup_{i \in I_e} V(C_i)$. Then P_e is the union of the vertex sets of paths in C such that all of them contain v_j and v_{j+1}, hence P_e is a set of consecutive vertices of C. By shifting the ordering of vertices of C, we may assume that $P_e = \{v_k, v_{k+1}, \ldots, v_l\}$, with $1 \leq k < l \leq n$. Thus there exists $i_1, i_2 \in I_e$ with $v_k \in V(C_{i_1})$ and $v_l \in V(C_{i_2})$. Consequently, $P_e = V(C_{i_1}) \cup V(C_{i_2})$, and by minimality of I, $|I_e| \leq 2$, and the claim is proved. Moreover, by minimality of I, for any $i, j \in I$, $V(C_i)$ is not a subset of $V(C_j)$. Therefore, one can define a natural ordering $C_1, C_2, \ldots, C_t$ of $\{C_i\}_{i \in I}$ such that every edge of C is contained either in exactly one of the C_i, or in exactly two consecutive cycles of $C_1, C_2, \ldots, C_t$. Thus by Lemma 3.5, there exists a subgraph K' with a nowhere-zero unoriented 4-flow such that $E(C) \subseteq E(K')$. □

4 Hamiltonian Graphs of Even Orders

In this section we assume that G is a hamiltonian graph of even order. If G is bipartite, then by Corollary A it admits a nowhere-zero unoriented 6-flow. So we may assume that G is not bipartite. Let $C = (v_1, \ldots, v_n, v_1)$ be a Hamilton cycle of G. There is an edge $v_i v_j$ such that $i - j$ is even, otherwise $\{v_i \mid i \text{ is even}\}$ and $\{v_i \mid i \text{ is odd}\}$ would be a bipartition of G, which leads to contradiction. With no loss of generality, assume that $e_1 = v_1 v_s$ is one such edge. Define

$$BE = E(C) \cup \{e_1\},\ IE = E(G) \backslash BE,\ CE = \{v_1v_2, v_2v_3, \ldots, v_{s-1}v_s, v_sv_1\}.$$

These labels stand for “boundary edges”, “inside edges” and “cycle edges”, respectively. Let $H = G[IE]$.

Lemma 4.1. *There exists an even Eulerian subgraph K_1 such that $IE \subseteq E(K_1)$.*

Proof. The proof is similar to the proof of Lemma 3.2. Let $1 \leq a_1 < a_2 < \cdots < a_{2m} \leq n, (m \geq 0)$ be the indices of odd vertices in H. For each $i = 1, \ldots, 2m-1$, let

$$E_i = \{v_{a_i}v_{a_i+1}, v_{a_i+1}v_{a_i+2}, \ldots, v_{a_{i+1}-1}v_{a_{i+1}}\},$$

and $A = E_1 \cup E_3 \cup \cdots \cup E_{2m-1}$. If H has no odd vertices, then simply define $A = \varnothing$.

If $|A| + |IE|$ is even, then the subgraph $K_1 = G[IE \cup A]$ is the desired subgraph. Otherwise, let K_1 be the subgraph induced by $IE \cup (CE \bigtriangleup A)$, where $\bigtriangleup$ denotes the symmetric difference of the two sets. Note that this union is edge-disjoint. For every vertex v_i, $1 \leq i \leq n$,

$$deg_{K_1}(v_i) \equiv deg_{CE}(v_i) + (deg_{IE}(v_i) + deg_A(v_i)) \equiv 0 \pmod 2.$$

So K_1 is Eulerian. In addition,

$$|IE \cup (CE \bigtriangleup A)| \equiv |IE| + |CE| + |A| \equiv |IE| + 1 + |A| \equiv 0 \pmod 2.$$

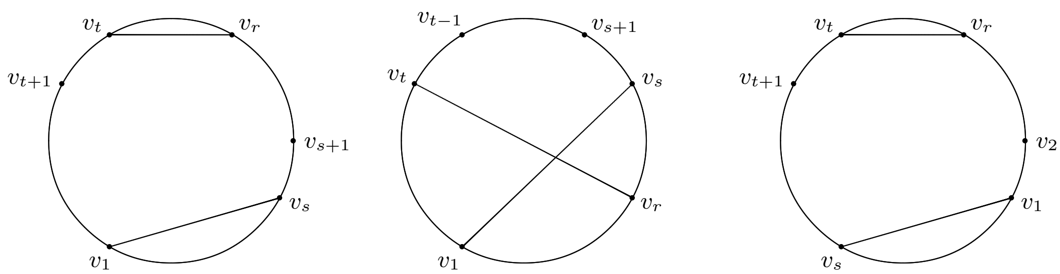
Hence K_1 is even, and the proof is complete. □

Lemma 4.2. *There exists a subgraph K of G that admits a nowhere-zero unoriented 3-flow and contains IE.*

Proof. The proof is similar to the proof of Lemma 3.3. By Lemma 4.1, there exists an even Eulerian subgraph K_1 containing IE. Consider odd and even components of K_1. Define functions f_1 and f_2 on $E(G)$ as we did in the proof of Lemma 3.3. Note that the number of odd components of K_1 and the order of G are even. Define $f = f_1 + f_2$ and let $K = G[\mathrm{supp}(f)]$. The function f is a nowhere-zero unoriented 3-flow in K, and we have $IE \subseteq E(K)$. □

Lemma 4.3. *There exists a subgraph K_2 that admits a nowhere-zero unoriented 4-flow and contains BE.*

Proof. Since G is a non-bipartite graph with a nowhere-zero unoriented flow, by Theorem A, $G \backslash \{v_1v_s\}$ is also non-bipartite. So there is another edge v_rv_t such that $t-r$ is even, and we may assume that $r < t$. We consider three cases, and in each of them we present two even cycles such that the



Case 1 Case 2 Case 3

Figure 3

union of their edges contains BE. By Lemma A, each of these even cycles admit a nowhere-zero unoriented 2-flow, thus according to Lemma 3.1, their union admits a nowhere-zero unoriented 4-flow. The cases are as follows (see Figure 3):

Case 1. $s < r < t$: The two cycles are $(v_1, v_s, v_{s+1}, \ldots, v_r, v_t, v_{t+1}, \ldots, v_n, v_1)$ and C.
Case 2. $r < s < t$: The two cycles are $(v_1, v_2, \ldots, v_r, v_t, v_{t-1}, \ldots, v_s, v_1)$ and C.
Case 3. $r < t < s$: The two cycles are $(v_1, v_2, \ldots, v_r, v_t, v_{t+1}, \ldots, v_s, v_1)$ and C. □

Theorem 4.1. *Let G be a hamiltonian graph of even order with a nowhere-zero unoriented flow. Then G admits a nowhere-zero unoriented 12-flow.*

Proof. By Lemmas 4.2 and 4.3, there exist two subgraphs K and K_2, with a nowhere-zero unoriented 3-flow and a nowhere-zero unoriented 4-flow, respectively, such that $IE \subseteq E(K)$ and $BE \subseteq E(K_2)$. Thus $E(G) \subseteq E(K) \cup E(K_2)$ and Lemma 3.1 completes the proof. □

Now, by Theorems 3.4 and 4.1, we are in a position to state our main result.

Theorem 4.2. *If G is a hamiltonian graph with a nowhere-zero unoriented flow, then G admits a nowhere-zero unoriented 12-flow.*

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