Cops and Robber with Fast Robber on Interval graphs and Chordal Graphs

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GrasCan'12

Notation

G the graph of the game, which is simple and connected.

n the number of vertices of G.

$c_\infty(G)$ the cop number of G.

What's Known?

- There are graphs with $c_\infty(G) = \Theta(n)$.
  [Frieze, Krivelevich, Loh'11]
- Computing $c_\infty(G)$ is NP-hard.
  [Fomin, Golovach, Kratochvíl'08]
- Computing $c_\infty(G)$ for an interval graph is in P. [Gavenčiak'11]

Today's Plan

1. Any interval graph has $c_\infty(G) = O(\sqrt{n})$ and this is best possible.
2. There are chordal graphs with $c_\infty(G) = \Omega(n/\log n)$.

In the usual game, both classes are cop-win.

Interval Graphs

Definition

Intersection graph of a set of closed intervals on the real line.

A Path Decomposition of a Graph

Definition

Let G be a graph, m be a positive integer, and $\{W_i : 1 \leq i \leq m\}$ be a family of subsets of $V(G)$, called the bags. The family $\{W_i\}$ is a path decomposition of G if it satisfies:

(i) $\cup_{1 \leq i \leq m} W_i = V(G)$.

(ii) For every $uv \in E(G)$, there is a bag containing both u and v.

(iii) For every $v \in V(G)$, v is contained in a consecutive set of bags.

Figure!

Wide Subgraphs

Definition

A subgraph H of G is k-wide if

(i) H is k-vertex-connected, and

(ii) No $k-1$ vertices of G dominate H.

Claim

If G has an k-wide subgraph H, then $c_\infty(G) \geq k$.

Proof.

The robber stays in H all the time! □

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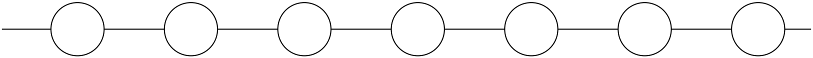

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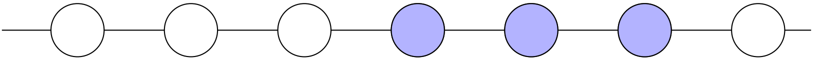

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The Upper Bound For Interval Graphs

Lemma

If M is the maximum number s. t. G has an M-wide interval subgraph,

$$M \leq c_\infty(G) \leq 3M$$

Proof.

For each subgraph H of G, at least one of the following holds:

(i) H has a cut set with M vertices.

(ii) There are M vertices of G that dominate H.

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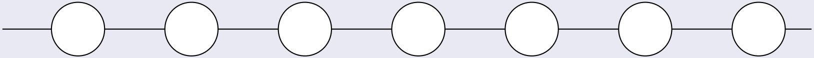

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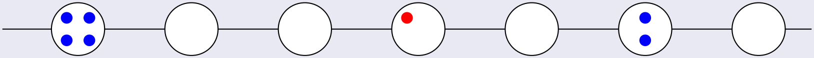

Invariant: Two teams keep the robber in the *interval subgraph*. □

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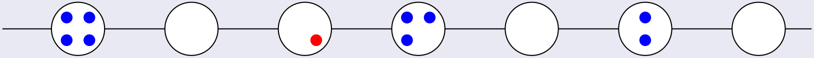

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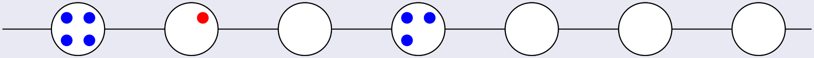

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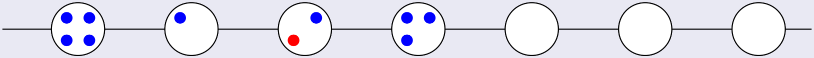

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Maximum Cop Number of Interval Graphs

Theorem

Let G be an interval graph. No interval subgraph of G is $(\sqrt{5n}+3)$*-wide.*

The theorem implies

$$c_\infty(G) = O(\sqrt{n})$$

The above graph is $\left(\frac{\sqrt{n}}{3}\right)$-wide.

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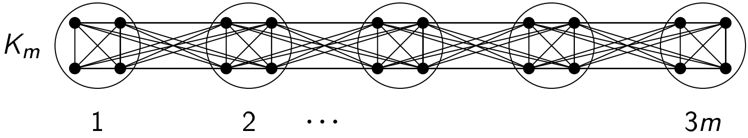

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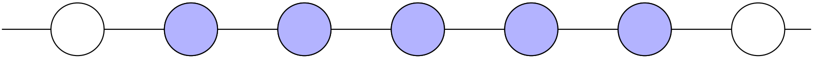

A: minimum dominating set for H

δ: minimum degree in A

$$|A|(\delta+1) \leq 5|V(H)| \leq 5n$$

If $|A| \leq \sqrt{5n}$ then H has small dominating set

If $\delta+1 \leq \sqrt{5n}$ then H has small cut set

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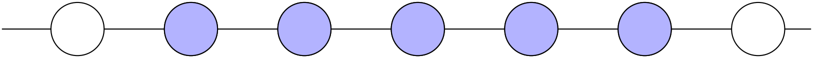

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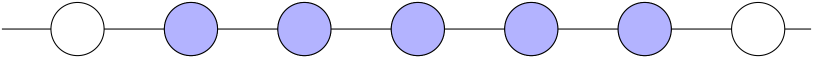

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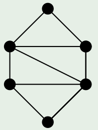
Chordal Graphs

Definition

No induced cycle with more than 3 vertices.

Example



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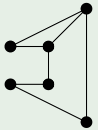

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Fact: Every interval graph is chordal.

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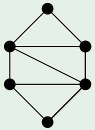

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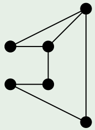

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Accessible Sets

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A subset $X \subseteq V(G)$ is called accessible if

- $c_\infty(G) \geq |X|$, and
- if there are $|X| - 1$ cops in the game, then there exists a strategy for the robber, in which the robber has access to X in every round.

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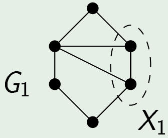
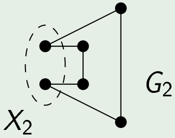
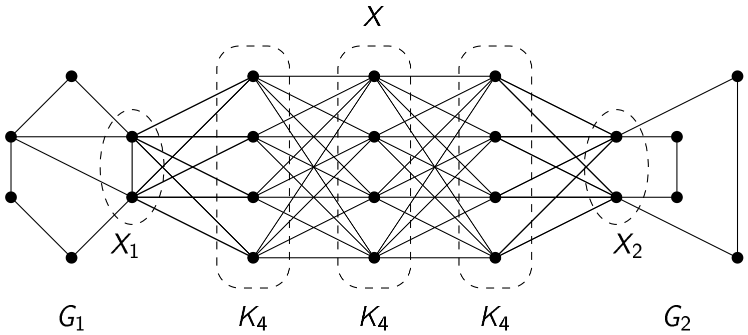




Construction of Chordal Graphs With Large Cop Number

X is accessible in this graph:



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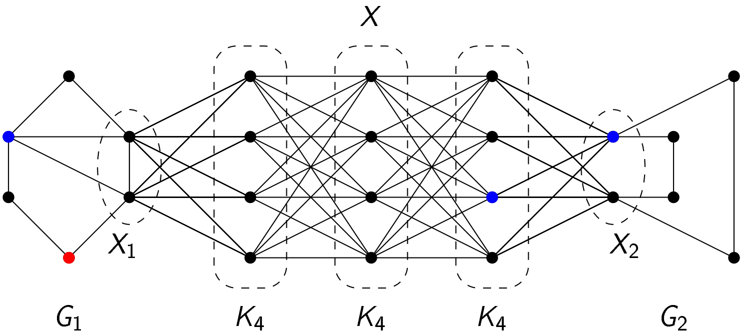

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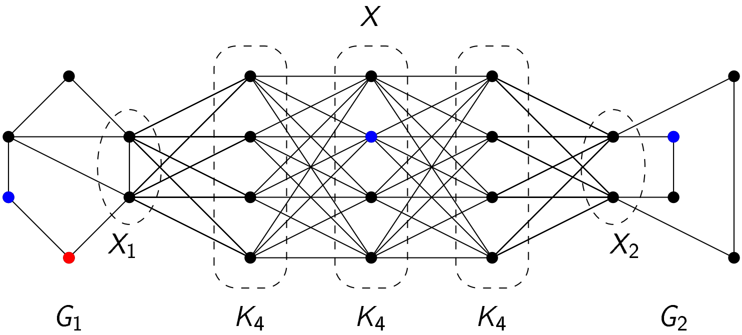

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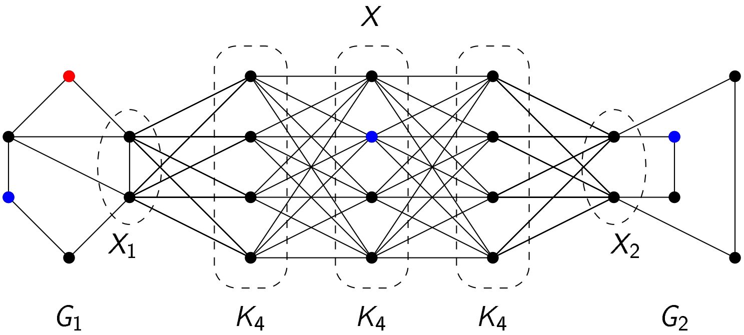

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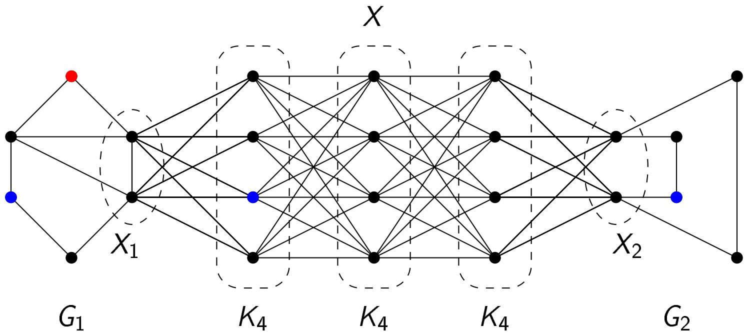

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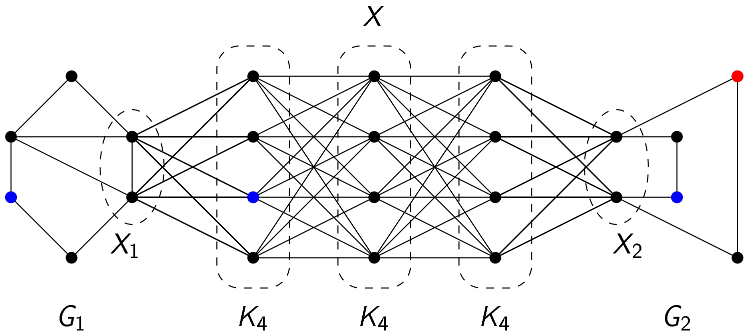

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Theorem

There exist chordal graphs with cop number $\Omega\left(\frac{n}{\log n}\right)$.

Proof.

Let $g(m)$ be the minimum order of a graph with an accessible subset of m vertices.

$$g(2m) \leq 2g(m) + 3 \times 2m$$

so $g(m) = O(m \log m)$. □

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An Open Problem

We proved that there are chordal graphs with

$$c_\infty(G) = \Omega(n/\log n) .$$

Is this bound tight? Are there chordal graphs with $c_\infty(G) = \Theta(n)$?

Thank You!

Any Questions?