It's a small world for random surfers

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joint work with Nick Wormald

The random-surfer Webgraph model

Model definition

- ✓ Parameters: p and d
- ✓ Consider a pool of independent Geo(p) random variables.
- ✓ Build a random graph with out-degree d: start with one vertex with d loops, add a new vertex in each step.

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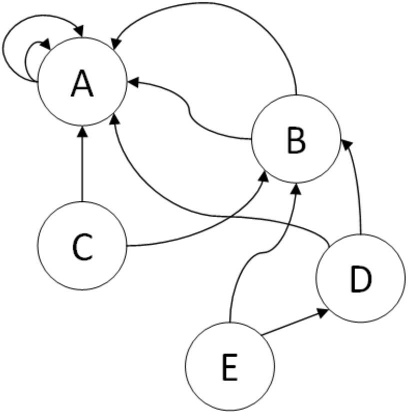
Say $p = 1/2$ and $d = 2$


A
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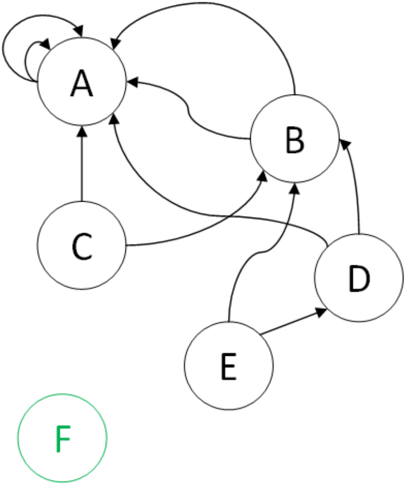
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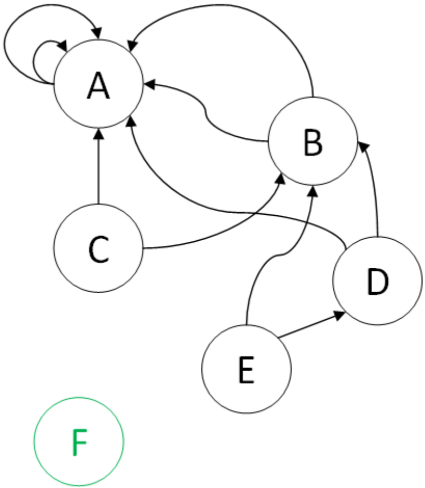
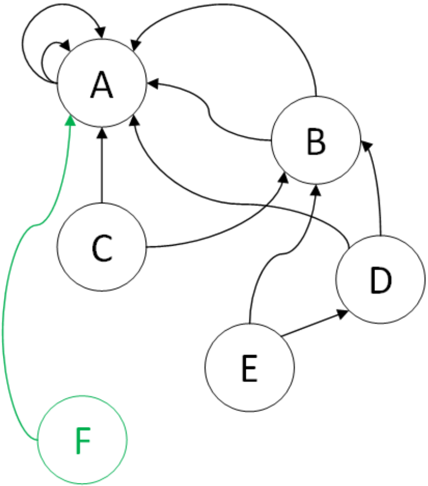
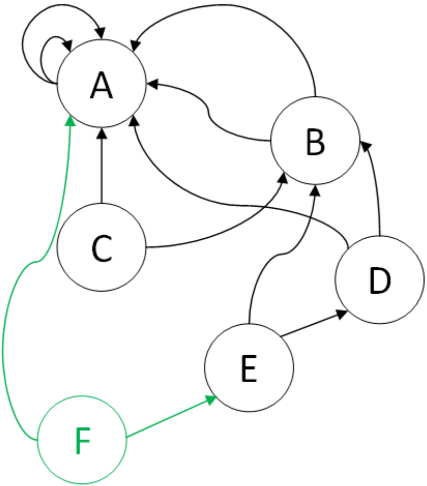


C, 2

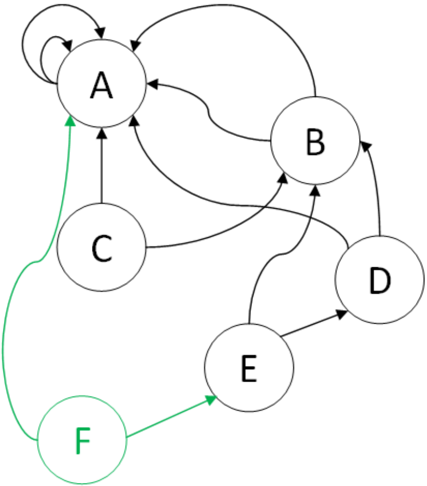


C, 2



C, 2

E, 0



C, 2

E, 0

Equivalently, PageRank-based selection

Previous work

- ✓ Pandurangan, Raghavan, Upfal'02: PageRank-based selection model definition, experimental results
- ✓ Blum, Chan, Rwebangira'06: random-surfer model definition, experimental results
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We studied the diameter of the underlying undirected graph...

Our result

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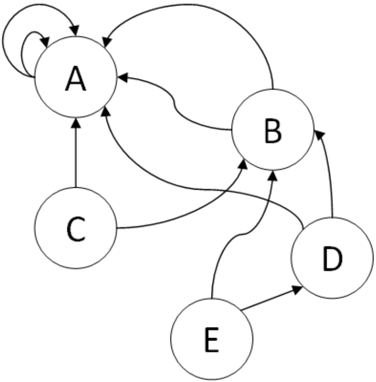
Proof idea: build a coupling with a weighted random recursive tree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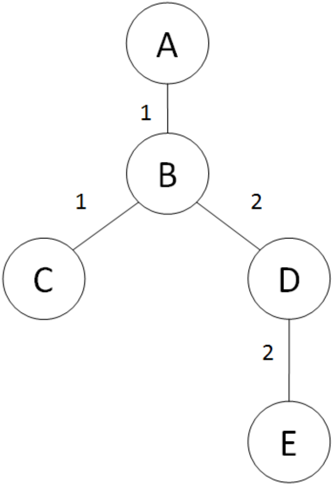
A
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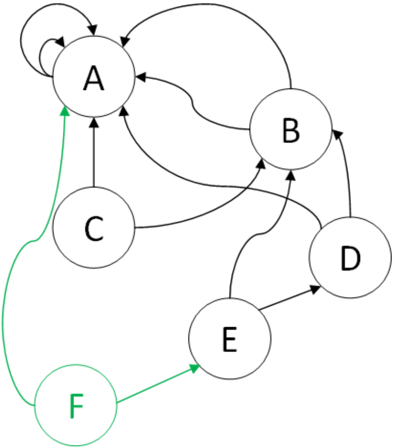
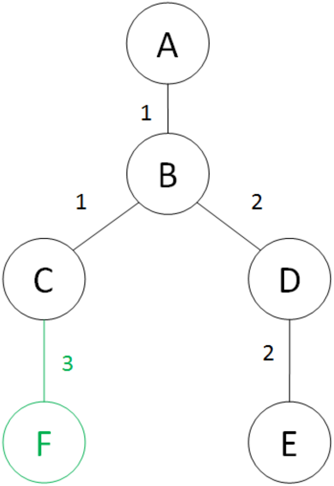
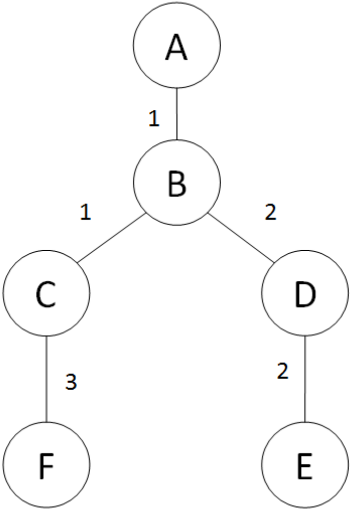


C, 2

E, 0

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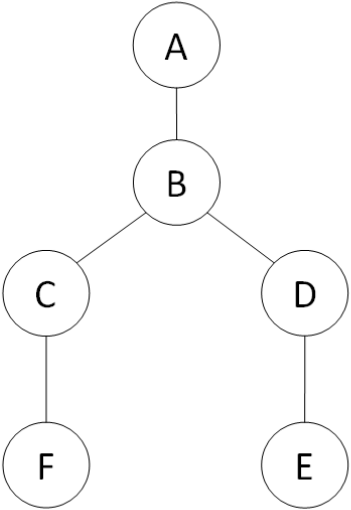
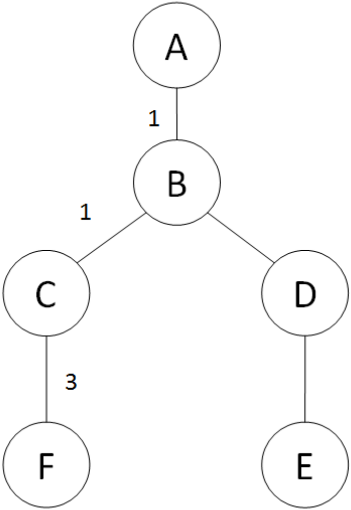
The height of the unweighted tree $\sim e \log n$ [Pittel'94]

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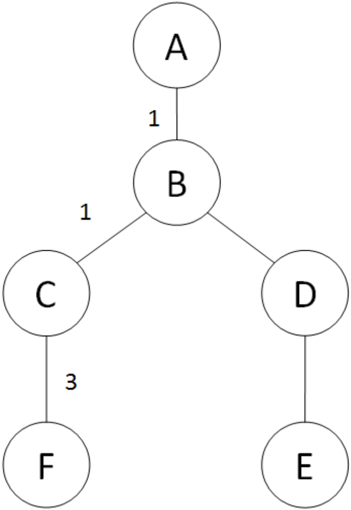
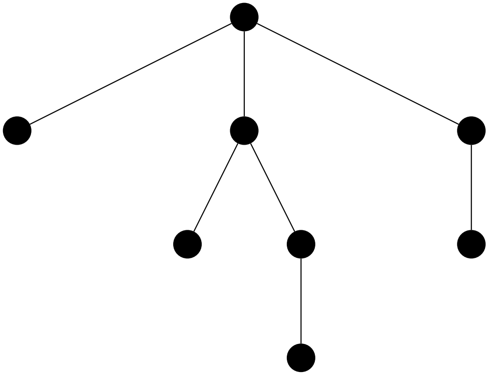
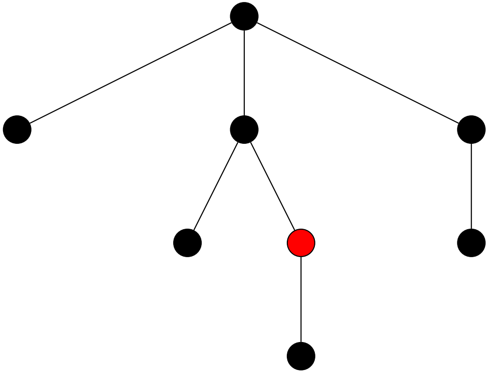


Theorem

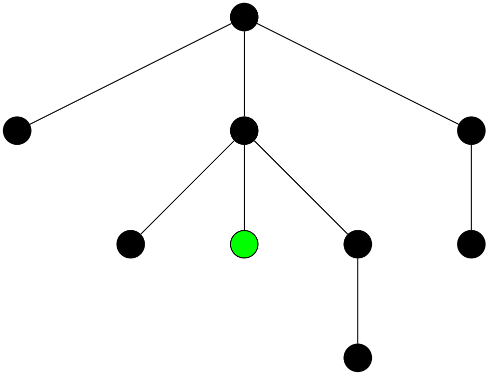
A.a.s. the height of the weighted tree $\leq (4e^p/p)\log n$

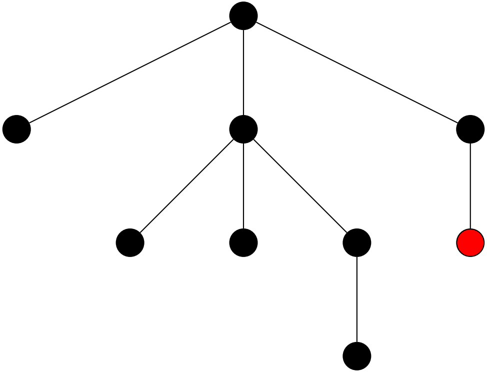
Random-surfer trees ($d = 1$)



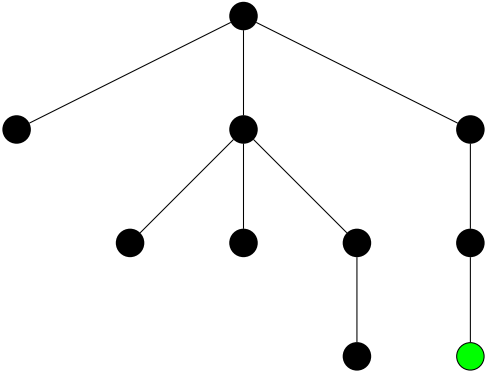


next geom.r.v. $= 1$





next geom.r.v. $= 0$



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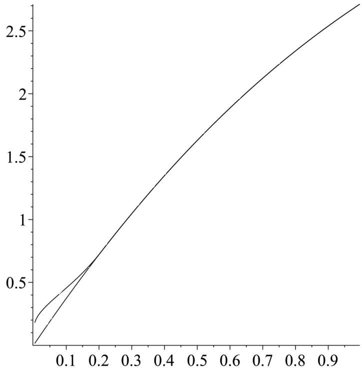
Theorem

A.a.s. the height is between $(L(p) - o(1)) \log n$ *and* $(U(p) + o(1)) \log n$*, and the diameter is between twice these values.*



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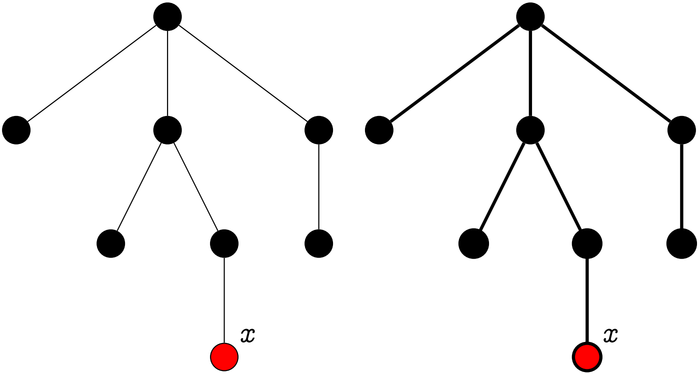
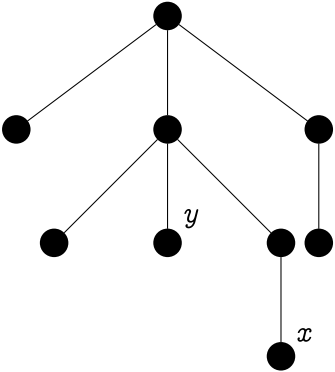
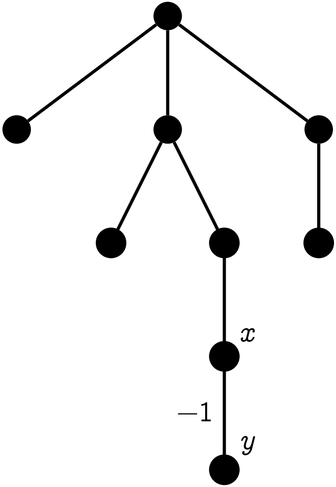
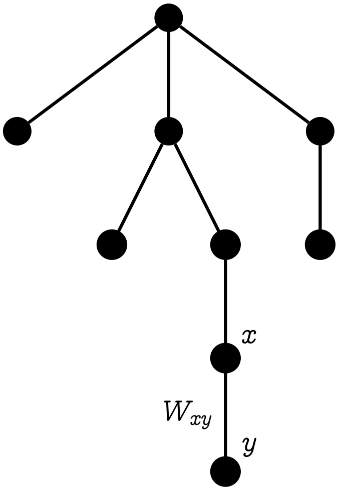
Proof idea: eliminate the complicated attachment rule by introducing weights,
then adapt a powerful technique of Broutin and Devroye'06 for analysing heights of weighted random trees



next geom.r.v. $= 2$ next geom.r.v. $= 2$

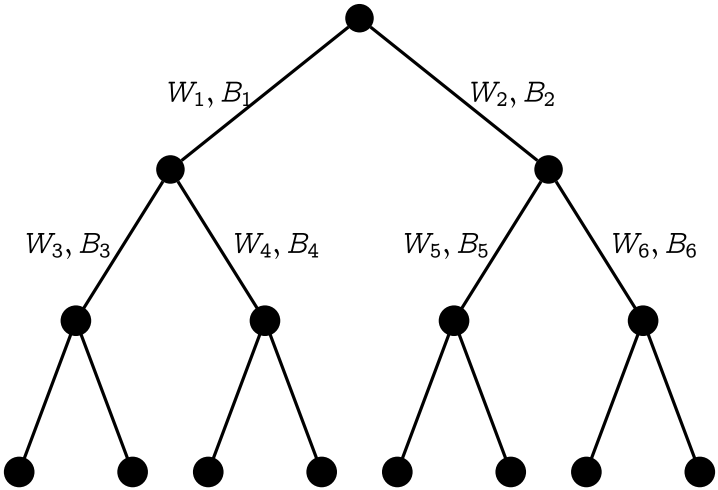

y
x


x
−1
y



$$W_{xy} = 1 - \text{Geo}(p)$$

The technique of Broutin and Devroye



Poissonization, and use the theory of large deviations

Full statement of our result

Given p and $\varepsilon > 0$, a.a.s. the height is between $(L(p) - \varepsilon)\log n$ and $(U(p) + \varepsilon)\log n$, and the diameter is between twice these values.

Let $p_0 \approx 0.206$ be the unique solution in $(0, 1/2)$ to

$$\log\left(\frac{1-p}{p}\right) = \frac{1-p}{1-2p} .$$

Let s be the solution in $(0, 1)$ to

$$s \log\left(\frac{(1-p)(2-s)}{1-s}\right) = 1 .$$

Then,

$$L(p) = \exp(1/s)s(2-s)p ,$$

and

$$U(p) = \begin{cases} L(p) & \text{if } p_0 \le p < 1 \\ \left(\log\left(\frac{1-p}{p}\right)\right)^{-1} & \text{if } 0 < p < p_0 . \end{cases}$$

Open problems

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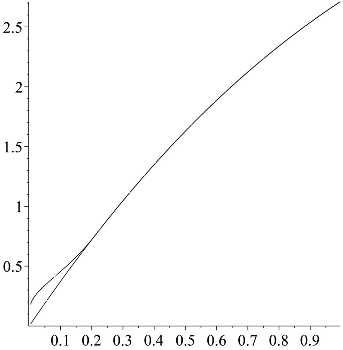
- ✓ Diameter of random-surfer tree is between $(2L(p) - o(1))\log n$ and $(2U(p) + o(1))\log n$



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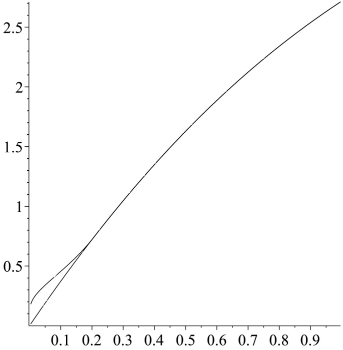

$$W_{xy} = \max\{1 - \text{Geo}(p), 1 - W_x\}$$